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Experimental investigation of universal parametric correlators using a vibrating plate

K. Schaadt[1,2] and A. Kudrolli[1]
[1]*Department of Physics, Clark University, Worcester, Massachusetts 01610*
[2]*Center for Chaos and Turbulence Studies, Niels Bohr Institute, Blegdamsvej 17, 2100 Copenhagen Ø, Denmark*
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The parametric variation of the eigenfrequencies of a chaotic plate is measured and compared to random matrix theory using recently calculated universal correlation functions. The sensitivity of the flexural modes of the plate to pressure is used to isolate this symmetry class of modes and simplify the data analysis. The size of the plate is used as the external parameter and the eigenvalues are observed to undergo one or two oscillations in the experimental window. The correlations of the eigenvalues are in good agreement with statistical measures such as the parametric number variance, the velocity autocorrelation, and the intralevel velocity autocorrelation derived for the Gaussian orthogonal ensemble of random matrix theory. Our results show that the theory can be also applied to wave systems other than quantum systems. [S1063-651X(99)51310-X]

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It has been widely recognized that the eigenvalues of a quantum system show universal features that depend only on the presence or absence of chaos in the corresponding classical or ray system [1–3]. The universality has been confirmed using not only quantum systems, but also systems that obey an elastomechanical wave equation [4–6]. The differences in the statistical properties has been recognized to be due to the presence of level repulsion which were demonstrated as avoided crossings as a system parameter was varied. However, it was postulated only recently that the resulting fluctuations of the energy levels also show universal properties that are independent of the nature of the parameter [7,8].

When a quantum system is subjected to a perturbation via an external parameter X, the eigenvalues change and oscillate as a function of X. Using supersymmetry techniques, Simons and Altshuler [7,8] were able to calculate the correlations as a function of external parameter for energy levels with Wigner-Dyson distributions of random matrix theory (RMT). The agreement of their analytical results with numerical simulations of disordered metallic rings and a chaotic billiard led them to the remarkable conjecture that correlations in the eigenvalues show universal features that are *independent* of the nature of the perturbation after appropriate normalization. Here the proper rescaling required to compare across different systems is given by expressing the energy E in units of the local mean level spacing Δ, and the parameter in units of the square root of the local mean squared slope:

$$\varepsilon = E/\Delta \quad x = \sqrt{\left\langle \left(\frac{d\varepsilon}{dX} \right)^2 \right\rangle} X, \tag{1}$$

where ε is the normalized energy and x is the rescaled external parameter.

The conjecture was tested further with numerical simulations of a hydrogen atom in a magnetic field, where agreement was found over a certain parameter range, but systematic deviations were also found because the system is only partially chaotic [9]. Although some of the correlations have been indirectly tested in the conductance fluctuations of electrons in ballistic cavities [10,11], and also in microwave cavities [12] and quartz blocks [13] where bouncing-ball-like modes complicate the analysis, there has been no report of a direct experimental test of their universality.

In this paper, we report direct experimental evidence of the universality of the above mentioned parametric correlators. A freely vibrating plate with the shape of a Sinai stadium [14] is used and the smooth motion of the eigenfrequencies is measured as a function of the size of the plate. Two classes of uncoupled modes exist in an isotropic plate: *flexural*, for which the displacement is perpendicular to the plane of the plate, and *in-plane*, for which the displacement is in the plane of the plate [15,17]. We are able to experimentally isolate the flexural modes and therefore can simplify the analysis by not having to consider problems associated with mixed symmetries. The flexural modes obey a scalar equation for the displacement W perpendicular to the plate:

$$(\nabla^2 - k^2)(\nabla^2 + k^2)W = 0, \tag{2}$$

where k denotes the wave number. The dispersion relation is given by

$$f = \frac{k^2}{2\pi}\sqrt{\frac{E_Y h^3}{12\rho(1-\nu^2)}}, \tag{3}$$

where f is the frequency, h is the thickness of the plate, ρ is the density, E_Y is Young's modulus, and ν is Poisson's ratio. Any solution W of Eq. (2) can be written as a superposition of two modes, W_1 and W_2, where

$$(\nabla^2 + k^2)W_1 = 0, \quad \text{and} \quad (\nabla^2 - k^2)W_2 = 0. \tag{4}$$

W_1 is a solution to the Helmholtz equation with free boundary conditions. W_2 is an *exponential* mode or *boundary* mode. The boundary modes are responsible for only about one percent of the density of states [16,17] and do not appear to alter the universality of the eigenvalues. Equation (2) is an approximation to the full elastomechanical wave equation in the limit where the wavelength is much larger than the thickness of the plate. The typical wavelength in our experiments

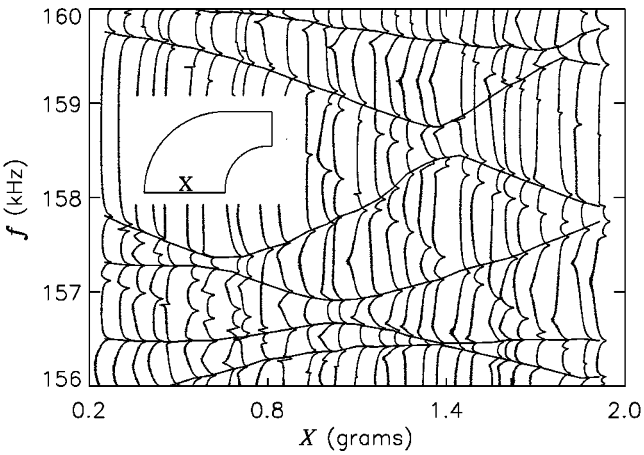

FIG. 1. Transmission amplitude as a function of the frequency in steps of the parameter X. The flexural modes are joined by a solid curve to guide the eye. Other modes pass through the diagram without any interaction with the flexural modes. These are the in-plane modes, which are not included in the data analysis (see text). Inset: Shape of the Sinai-stadium plate. The side which is polished to effect a parametric change is indicated by X.

is 8 mm, and the thickness of the plate is 2 mm. In this case, Eq. (2) is a good approximation.

We compare statistical observables of the eigenvalue motion as a function of the parameter to analytical calculations. In particular, we find that the data agree with calculations of the parametric number variance $v(x)$ by Simons *et al.* [9] and shows a linear behavior for small x which is different from semiclassical calculations [18]. To investigate correlations in energy-parameter space, comparisons are made with theoretical calculations for the velocity autocorrelation $c(x)$ and the intralevel velocity autocorrelation $\tilde{c}(w,x)$, which describes the correlations between the rate of change of eigenvalues separated in energy by w and in parameter by x [8]. Good agreement is observed for selected values of w and overall x. Combined, these results provide experimental evidence of the universality of a broad class of the statistical observables of parametric level motion that have been studied theoretically.

In the experiments we use an aluminum plate of thickness 2.0 mm, machined in the shape of a quarter Sinai stadium with radii 40 mm and 70 mm (see Fig. 1). The plate rests on three piezoelectric transducers, of which one is a transmitter and two are receivers. We measure acoustic transmission spectra of the plate using a HP 4395A network analyzer. A

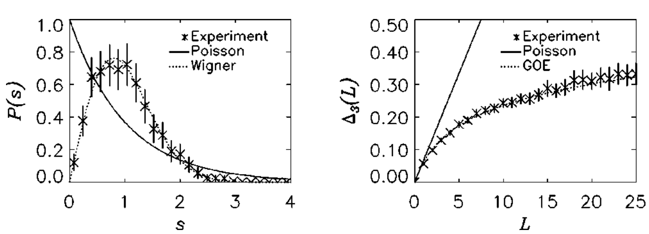

FIG. 2. Left: Distribution of nearest neighbor spacings $P(s)$ from all 63 measured spectra (crosses), compared to the Wigner distribution of RMT (solid curve). Right: Spectral rigidity $\Delta_3(L)$ for the experimental data (crosses) compared to the GOE result (solid curve).

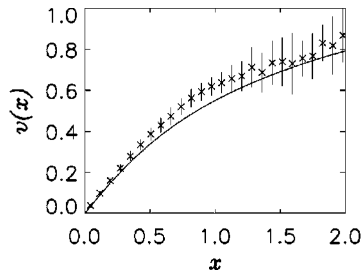

FIG. 3. Comparison of experimental data (crosses) and RMT (solid curve) for the parametric number variance $v(x)$.

sample of the transmission signal at different values of the parameter is shown in Fig. 1. The amplitudes of the resonances depend on the location of the transducers but the eigenfrequencies are unchanged. The plate is kept in a temperature-controlled oven held at 300 K to within 1 mK. A vacuum system ensures that the air pressure is below 10^{-1} Torr, which is low enough that air damping of the plate is insignificant compared to other damping mechanisms. Of the two classes of modes, flexural modes are more sensitive to the presence of air damping than in-plane modes because of the flexural out-of-plane oscillation. We find that going from vacuum to atmospheric pressure, the Q factor of the flexural modes decreases by at least a factor of 3, whereas the Q factor for the in-plane modes is unchanged.

We first measure the transmission spectrum of the plate, then decrease the size of the plate by sanding off material at the longest straight edge, as indicated in Fig. 1. The amount of material removed is determined by measuring the mass of the plate to within 5×10^{-5} grams. Approximately 5×10^{-2} grams is removed each time and in all 6% of the material is removed in 63 steps. The spectrum is measured in the interval between 100 kHz and 300 kHz. Periodically, the spectrum is also measured at 1 atm to identify the flexural modes. After this separation, we find approximately 300 resonances, of which 25 drift out of the frequency window due to the overall increase in frequency when the size of the plate is decreased. A resonance frequency can be determined to within 0.5 Hz by fitting the resonance peak to a Breit-Wigner function. We are confident that all eigenfrequencies in the frequency window are detected, because it is impossible for the amplitude of a resonance peak to lie below our detection level for all 63 values of the parameter. The absence of interaction of the flexural modes with the in-plane modes is checked to within experimental accuracy by noting a lack of interaction at flexural in-plane encounters (see Fig. 1).

In the data analysis, the implementation of the normalization or *unfolding* given by Eq. (1) is of great importance. Since the cumulative level density or *staircase function* for a freely vibrating plate was recently calculated [16], both the mean level spacing and the mean squared velocity are known analytically. This knowledge can be directly applied to our data, which makes the data analysis very clean from a theoretical viewpoint.

Traditional statistical quantities such as the nearest neighbor energy spacings $P(s)$, which measures short range correlations, and spectral rigidity $\Delta_3(L)$, which measures the long range correlations, [19] are first used to test if the system belongs to the Gaussian orthogonal ensemble (GOE)

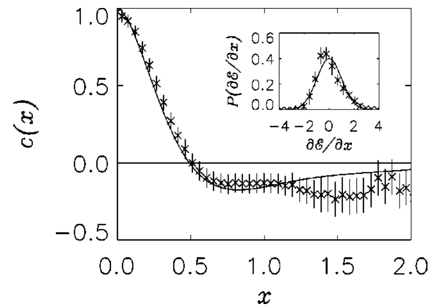

FIG. 4. Velocity autocorrelation $c(x)$ (crosses) compared to RMT calculations (solid curve). Inset: Distribution of eigenvalue velocities compared to a Gaussian distribution.

universality class. Here s and L correspond to the range of energy normalized by the mean energy level spacing over which the correlation is calculated. The data is shown in Fig. 2 and complete agreement with GOE is observed. Fully chaotic systems are very rare and most chaotic geometries have regions in phase space which are integrable. The Sinai stadium geometry is no exception and is known to have small regions of integrability. However, if these regions are very small, they can support an integrable level only at very high frequencies, and therefore complete agreement with GOE is expected and observed.

We now present the main results, which consists of the correlations in the parametric variation of the eigenfrequencies. The parametric number variance $v(x)$ is defined as

$$v(x)=\langle[n(\varepsilon,x')-n(\varepsilon,x'+x)]^2\rangle, \tag{5}$$

where the average is over the parameter x' and energy ε. Here, $n(\varepsilon,x)$ is the staircase function which counts the number of energy levels at fixed x with energy lower than ε. The parameter x has been normalized according to Eq. (1), as explained above. The variance measures the difference in the number of eigenvalues which are below a fixed value of normalized energy ε. Therefore, this quantity measures the collective motion of levels under parametric change [18]. Comparison of the data with theory is shown in Fig. 3. The $v(x)$ calculated from the data grows linearly from zero and has a slope of 0.8 ± 0.01, which is in excellent agreement with the calculated value of $\sqrt{2/\pi}\sim0.797$ by Simons *et al.* [9]. A saturation is expected at large values of x and therefore the $v(x)$ becomes sublinear at higher x.

To investigate the oscillations of the eigenvalues with the parameter x, a new set of measures are required that study the rate of change of the eigenvalue as a function of parameter [7]. One example is the *intralevel velocity autocorrelation* $\tilde{c}(\omega,x)$, which correlates velocities separated by a distance x in parameter space and by a distance ω in energy:

$$\tilde{c}(\omega,x)=\frac{\sum_{n,m}\left\langle\delta[\varepsilon_n(x')-\varepsilon_m(x'+x)-\omega]\frac{\partial\varepsilon_n(x')}{\partial x'}\frac{\partial\varepsilon_m(x'+x)}{\partial x'}\right\rangle}{\sum_{n,m}\langle\delta[\varepsilon_n(x')-\varepsilon_m(x'+x)-\omega]\rangle} \tag{6}$$

The average is over the parameter x'. Using the supersymmetric nonlinear σ model developed by Efetov [20], Simons and Altshuler derived an integral representation for the intralevel velocity autocorrelation. Another correlation is the *velocity autocorrelation* $c(x)$, which correlates velocities that belong to the same energy level:

$$c(x)=\left\langle\frac{\partial\varepsilon(x')}{\partial x'}\frac{\partial\varepsilon(x'+x)}{\partial x'}\right\rangle. \tag{7}$$

The brackets denote an average over the parameter x' and the energy ε. For this correlator no analytical results exist for intermediate values of x. Therefore, we compare our result for $c(x)$ to a curve calculated by Mucciolo [21] using large GOE matrices that agree with the analytical results in the limit of large and small x.

We first present the result for the velocity autocorrelation $c(x)$ (see Fig. 4). For values of x smaller than 1, we find good agreement with the numerical RMT curve [21]. At larger values of x; however, we see a deviation which is outside the experimental error bars. The shape of the correlation function indicates that the slope $\partial\varepsilon(x)/\partial x$ changes smoothly and has opposite signs near $x=0.5$ because the parameter x has been normalized to correspond to approximately one oscillation for $x=1$. This behavior of the correlation functions indicates that, locally, there is a particular length scale over which eigenfrequencies oscillate. The distribution of velocities $\partial\varepsilon(x)/\partial x$ of the eigenvalues should be a Gaussian with a mean value of zero. The data is shown in the inset of Fig. 4. The data is close to a Gaussian, but is slightly asymmetric with more velocities of small magnitude that are negative than positive. We emphasize that the mean slope is zero, indicating that this discrepancy does not originate in the normalization of the eigenfrequencies. We believe that the deviation is due to a finite data set. It appears that the correlations are very robust and give good agreement even if the velocity distribution is not exactly Gaussian.

To make a more stringent test of the correlations, we compare our data with the intralevel velocity autocorrelation $\tilde{c}(\omega,x)$ for $\omega=0.25$, $\omega=0.50$, and $\omega=1.0$ as shown in Fig. 5. We compare our data to a numerical evaluation of the integral representation of this correlator [8]. In calculating these quantities we have averaged over a small energy window of $\delta\omega=0.03$ which is also done in the theoretical calculations. The occurrence of the peaks in the correlation functions and the systematic increase of the value of x where

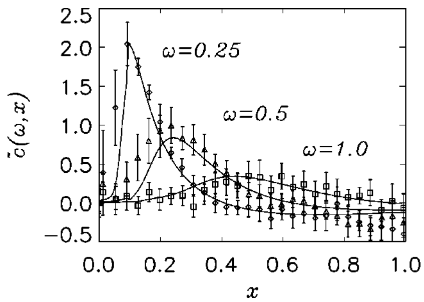

FIG. 5. Intralevel velocity autocorrelation for $\omega = 0.25$ (diamonds), $\omega = 0.5$ (triangles), and $\omega = 1.0$ (squares). The theory curves correspond to analytical calculation of Simons and Altshuler [8] using supersymmetry techniques.

the peak occurs can be understood from the fact that near an avoided crossing, one has to go across by nearly as much along the normalized energy axis as along the parameter axis to encounter a similar slope (see Fig. 1). Comparison of the data in Fig. 5 shows very good agreement for all three values of ω, validating the theory.

In conclusion, we have investigated experimentally the parametric-level motion of the flexural modes of a freely vibrating plate as a function of the size of the plate. We have used our data to calculate statistical quantities that probe the parametric motion of the levels, and found agreement with the universal predictions of RMT. The agreement with RMT suggests that the universal predictions for parametric-level motion extends beyond quantum chaotic systems to a wider range of wave systems, including acoustical waves.

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